Prime Theorem 2

We use the notation $a \mid b$ to mean "a divides b," and $a \nmid b$ to mean "a does not divide b," For example, $3 \mid 6$ but $3 \nmid 7$. Two positive integers m and n are called *relatively prime* if they have no common factor than 1. Recall $\mathcal{P}$ denotes the set of all prime numbers.

We show

Given two positive integers m and n selected independently, the probability that m and n are relatively prime is $6/\pi^2$.

A step in this is to sum the series $\sum_{n=1}^{\infty} 1/n^2$. This is a p-series with $p = 2$, so we know it converges. We do not – so far – know to what it converges.

We compute the probability that m and n are relatively prime,

$$\begin{aligned}
&P(m \text{ and } n \text{ are relatively prime})\\
&= P(\text{ for all } p \in \mathcal{P}, p \nmid m \text{ or } p \nmid n) \quad \text{no common factor}\\
&= P\left(\bigcap_{p \in \mathcal{P}} (p \nmid m \text{ or } p \nmid n)\right)\\
&= \prod_{p \in \mathcal{P}} P(p \nmid m \text{ or } p \nmid n) \quad \text{Prob of the intersection is the product of the probs.}\\
&= \prod_{p \in \mathcal{P}} (1 - P(p \mid m \text{ and } p \mid n)) \quad \text{not A or not B is not (A and B)}\\
&= \prod_{p \in \mathcal{P}} (1 - P(p \mid m) \cdot P(p \mid n)) \quad \text{This uses independence.}\\
&= \prod_{p \in \mathcal{P}} \left(1 - \frac{1}{p} \cdot \frac{1}{p}\right) \quad \text{The probability 2 divides } m \text{ is } \frac{1}{2}, \text{ 3 divides } m \text{ is } \frac{1}{3}, \text{ etc.}\\
&= \left(\prod_{p \in \mathcal{P}} \frac{1}{1 - \frac{1}{p^2}}\right)^{-1} \quad \text{The reciprocal of the product is the product of the reciprocals.}\\
&= \left(\prod_{p \in \mathcal{P}} \sum_{n=0}^{\infty} \left(\frac{1}{p^2}\right)^n\right)^{-1} \quad \text{Recognize the sum of a geometric series.}\\
&= \left(\sum \frac{1}{(p_1^{n_1} \cdots p_l^{n_l})^2}\right)^{-1} \quad \text{This is a bit complicated. See *.}
\end{aligned}$$

$$= \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{-1} \quad \text{prime decomposition of every integer} > 1.$$

To evaluate this sum, we use a trick that appears to come from nowhere, or at least from nowhere polite. Our study of series is based on approximating functions by polynomials. This works fine, but we have seen it is not so efficient for periodic functions. For x not close to 0, we must use a lot of terms from the Taylor series for $\sin(x)$ to calculate a reasonably accurate approximation of $\sin(x)$. Because many natural phenomena, sounds for example, are periodic signals, another approach is to approximate functions with sums of sines or sums of cosines. Odd functions (for all x in the domain, $f(-x) = -f(x)$) can be approximated by sums of sines, even functions ($f(-x) = f(x)$) by sums of cosines. Functions that are neither odd nor even require sums of both sines and cosines. These are called *Fourier series* representations of the functions.

Consider the function $f(x) = (\pi - |x|)^2$ (obtained from the same not polite location), with domain $[-\pi, \pi]$. This is an even function, and can be written as

$$f(x) = \sum_{n=0}^{\infty} c_n \cos(nx)$$

Assuming this is true, how can we find the coefficients c_n?

Because

$$\int_{-\pi}^{\pi} \cos(nx)dx = 0 \text{ for } n \neq 0,$$

we see

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} c_0 dx = 2\pi c_0.$$

That is,

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx.$$

Next, for integers $m \neq n$,

$$\begin{aligned} &\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx \\ &= \frac{(m+n)\sin((m-n)\pi) + (m-n)\sin((m+n)\pi)}{m^2 - n^2} = 0 \end{aligned}$$

and

$$\int_{-\pi}^{\pi} \cos^2(mx)dx = \pi$$

Then for $m > 0$,

$$\begin{aligned} \int_{-\pi}^{\pi} f(x)\cos(mx)dx &= \int_{-\pi}^{\pi} \left(\sum_{n=0}^{\infty} c_n \cos(nx)\right)\cos(mx)dx \\ &= \sum_{n=0}^{\infty} c_n \left(\int_{-\pi}^{\pi} \cos(nx)\cos(mx)dx\right) \\ &= c_m \pi \end{aligned}$$

That is, for $m > 0$,

$$c_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) dx.$$

For the function $f(x) = (\pi - |x|)^2$ we have

$$\begin{aligned} c_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\ &= \frac{1}{2\pi} \left(\int_{-\pi}^{0} (\pi - (-x))^2 dx + \int_{0}^{\pi} (\pi - x)^2 dx \right) \\ &= \frac{\pi^2}{3} \end{aligned}$$

and

$$\begin{aligned} c_m &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(mx) dx \\ &= \frac{1}{2\pi} \left(\int_{-\pi}^{0} (\pi - (-x))^2 \cos(mx) dx + \int_{0}^{\pi} (\pi - x)^2 \cos(mx) dx \right) \\ &= \frac{4}{m^2} \end{aligned}$$

Using these values for the coefficients, we see

$$\begin{aligned} \pi^2 = f(0) &= \sum_{n=0}^{\infty} c_n \cos(0) \\ &= c_0 + 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \end{aligned}$$

This gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and so

$$P(m \text{ and } n \text{ are relatively prime}) = \frac{6}{\pi^2}$$

* Writing $\mathcal{P} = \{p_1, p_2, p_3, \dots\}$, we expand the factors of

$$\prod_{p \in \mathcal{P}} \sum_{n=0}^{\infty} \left(\frac{1}{p^2} \right)^n$$

as

$$
\begin{aligned}
&\left(1+\frac{1}{p_1^2}+\left(\frac{1}{p_1^2}\right)^2+\left(\frac{1}{p_1^2}\right)^3+\left(\frac{1}{p_1^2}\right)^4+\cdots\right)\\
&\cdot\left(1+\frac{1}{p_2^2}+\left(\frac{1}{p_2^2}\right)^2+\left(\frac{1}{p_2^2}\right)^3+\left(\frac{1}{p_2^2}\right)^4+\cdots\right)\\
&\cdot\left(1+\frac{1}{p_3^2}+\left(\frac{1}{p_3^2}\right)^2+\left(\frac{1}{p_3^2}\right)^3+\left(\frac{1}{p_3^2}\right)^4+\cdots\right)\\
&\qquad\qquad\cdots
\end{aligned}
$$

This consists of the sum of all products of terms, one from each factor. Note if infinitely many of the terms are not 1, that product is 0. Thus the only non-zero terms have the form $1/(p_{i_1}\cdot p_{i_2}\cdots p_{i_k})^2$. Some of these primes may be repeated, so we obtain the next line of the equality.